Tutorial Notes 4

1. Find the volume of the region enclosed by the cone $\phi = \pi/3$, the xy-plane and the sphere $\rho = 2$.

   Solutions:

   The region is $0 \le \rho \le 2$, $\pi/3 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$. Hence the volume is

$$\begin{aligned}
&\int_{\pi/3}^{\pi/2} \int_0^2 \int_0^{2\pi} \rho^2 \sin\phi \, \mathrm{d}\theta \, \mathrm{d}\rho \, \mathrm{d}\phi \\
&= 2\pi \int_{\pi/3}^{\pi/2} \int_0^2 \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \\
&= 2\pi \cdot \frac{8}{3} \cdot \int_{\pi/3}^{\pi/2} \sin\phi \, \mathrm{d}\phi \\
&= \frac{8\pi}{3}.
\end{aligned}$$

2. Find the volume of the region enclosed by the paraboloids $z = 5 - x^2 - y^2$ and $z = 4x^2 + 4y^2$.

   Solutions:

   The region is $4x^2 + 4y^2 \le z \le 5 - x^2 - y^2$. In cylindrical coordinates, the region is $4r^2 \le z \le 5 - r^2$. Hence the volume is

$$\begin{aligned}
&\int_0^1 \int_{4r^2}^{5-r^2} \int_0^{2\pi} r \, \mathrm{d}\theta \, \mathrm{d}z \, \mathrm{d}r \\
&= 2\pi \int_0^1 r(5 - 5r^2) \, \mathrm{d}r \\
&= \frac{5\pi}{2}.
\end{aligned}$$

3. Find the moment of inertia about the z-axis of the ball $\rho \le a$ if the density δ is ρ^2 or $\rho \sin\phi$.

   Solutions:

   If $\delta = \rho^2$, the moment of inertia is

$$\begin{aligned}
&\int_0^{\pi} \int_0^a \int_0^{2\pi} \rho^2 \cdot (\rho \sin\phi)^2 \cdot \rho^2 \sin\phi \, \mathrm{d}\theta \, \mathrm{d}\rho \, \mathrm{d}\phi \\
&= 2\pi \int_0^{\pi} \int_0^a \rho^6 \sin^3\phi \, \mathrm{d}\rho \, \mathrm{d}\phi
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi \cdot \frac{a^7}{7} \cdot \int_0^{\pi} \sin^3\phi \,\mathrm{d}\phi \\
&= 2\pi \cdot \frac{a^7}{7} \cdot \int_0^{\pi} (1-\cos^2\phi)\,\mathrm{d}(-\cos\phi) \\
&= \frac{8\pi a^7}{21}.
\end{aligned}
$$

If $\delta = \rho\sin\phi$, the moment of inertia is

$$
\begin{aligned}
&\int_0^{\pi}\int_0^{a}\int_0^{2\pi} \rho\sin\phi \cdot (\rho\sin\phi)^2 \cdot \rho^2\sin\phi \,\mathrm{d}\theta\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&= 2\pi \int_0^{\pi}\int_0^{a} \rho^5 \sin^4\phi \,\mathrm{d}\rho\,\mathrm{d}\phi \\
&= \frac{\pi a^6}{3}\int_0^{\pi} \sin^4\phi\,\mathrm{d}\phi \\
&= \frac{\pi a^6}{3}\int_0^{\pi} \frac{(1-\cos 2\phi)^2}{4}\,\mathrm{d}\phi \\
&= \frac{\pi a^6}{3}\int_0^{\pi} \frac{1-2\cos 2\phi+\cos^2 2\phi}{4}\,\mathrm{d}\phi \\
&= \frac{\pi a^6}{3}\int_0^{\pi} \left(\frac{1-2\cos 2\phi}{4}+\frac{1+\cos 4\phi}{8}\right)\mathrm{d}\phi \\
&= \frac{\pi a^6}{3}\cdot\frac{3\pi}{8} \\
&= \frac{\pi^2 a^6}{8}.
\end{aligned}
$$

4. Evaluate

$$\int_{\Omega} (3x^2+14xy+8y^2)\,\mathrm{d}x\,\mathrm{d}y,$$

where Ω is the region enclosed by $y=(-3/2)x+1$, $y=(-3/2)x+3$, $y=(-1/4)x$, $y=(-1/4)x+1$.

Solutions:

Consider $u = 3x+2y$ and $v = x+4y$. Then

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = 10.$$

Moreover, note that

$$3x^2+14xy+8y^2 = (3x+2y)(x+4y).$$

By the change of variable formula,

$$\int_{\Omega}(3x^2+14xy+8y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_{\widetilde{\Omega}} uv\frac{\mathrm{d}u\,\mathrm{d}v}{10},$$

where $\widetilde{\Omega}$ is $2 \le u \le 6$, $0 \le v \le 4$. Then

$$\int_{\widetilde{\Omega}} uv\frac{\mathrm{d}u\,\mathrm{d}v}{10} = \int_2^6\int_0^4 \frac{uv}{10}\,\mathrm{d}v\,\mathrm{d}u = \frac{64}{5}.$$

5. Evaluate

$$\int_1^2 \int_{1/y}^{y} (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y.$$

Solutions:

The region is $1/y \le x \le y$ if $1 \le y \le 2$, $y/4 \le x \le 4/y$ if $2 \le y \le 4$, which is equivalent to $1 \le xy \le 4$, $1 \le y/x \le 4$, $x, y > 0$. Consider $u = xy$ and $v = y/x$, then

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \frac{2y}{x} = 2v.$$

By the change of variable formula,

$$\int_1^2 \int_{1/y}^{y} (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_{\widetilde{\Omega}} \left(\frac{u}{v} + uv\right) \frac{\mathrm{d}u\,\mathrm{d}v}{2v},$$

where $\widetilde{\Omega}$ is $1 \le u \le 4$, $1 \le v \le 4$. Then

$$\int_{\widetilde{\Omega}} \left(\frac{u}{v} + uv\right) \frac{\mathrm{d}u\,\mathrm{d}v}{2v} = \int_1^4 \int_1^4 \left(\frac{u}{2v^2} + \frac{u}{2}\right) \mathrm{d}u\,\mathrm{d}v = \frac{45}{16} + \frac{45}{4} = \frac{225}{16}.$$